PreCal_concise

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Here is the more concise summary of the main content, including definitions and examples, from “PreCalc.pdf”, in English and Markdown format, with shortened narrative parts:

1 | FUNCTIONS AND GRAPHS

This chapter reviews fundamental functions (polynomial, rational, trigonometric, exponential, logarithmic) for calculus study. It covers evaluation, graphical properties, and algebraic techniques for solving related equations, providing a foundation for calculus. For instance, logarithmic functions are used to compare earthquake intensity (Example 1.39).

1.1 | Review of Functions

This section defines functions and examines representations (tables, formulas, graphs). It covers notation, domain, range, function composition, and symmetry.

- **Function**
  - **Definition**: A function consists of a set of inputs (the **domain**), a set of outputs (the **range**), and a rule assigning each input to exactly one output.
  - The input is the **independent variable** (x), and the output is the **dependent variable** (y).
  - **Example**: For $f(x) = x^2$:
    * Domain: all real numbers.
    * Rule: square the input.
    * Input $x = 2$ gives output $y = 4$.
    * Range: non-negative real numbers.
  - Functions can be visualized as an input/output device (Figure 1.2) or by mapping elements (Figure 1.3), or by plotting points $(x, f(x))$ (Figure 1.4, Figure 1.5).
- **Domain of a Function**
  - If not specified, the domain includes all real numbers for which $f(x)$ is real.
  - **Example**: $f(x) = x^2$ domain is all real numbers; $f(x) = \sqrt{x}$ domain is non-negative real numbers.
  - **Set Notation**: Use set-builder (e.g., $\{x \in \mathbb{R} | 1 < x < 5\}$) or interval notation (e.g., $(1, 5)$) for infinite sets. Square brackets include endpoints (e.g., $$). ∞ and $-\infty$ denote unboundedness.
- **Piecewise-Defined Functions**
  - Functions defined by different equations across different domain parts.
  - **Example**: $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ -x+7 & \text{if } x \geq 2 \end{cases}$. $f(0) = 1$, $f(4) = 3$.
- **Evaluating Functions**
  - Substitute x into the formula.
  - **Example 1.1**: For $f(x) = x^2 - 3x + 5$:
    * $f(1) = 1^2 - 3(1) + 5 = 3$.
    * $f(-2) = (-2)^2 - 3(-2) + 5 = 15$.
    * $f(a) = a^2 - 3a + 5$.
- **Finding Domain and Range**
  - **Example 1.2**:
    * For $f(x) = \frac{1}{x+2}$: domain is $(-\infty, -2) \cup (-2, \infty)$; range is $(-\infty, 0) \cup (0, \infty)$.

    - For $f(x) = \sqrt{x-3}$: domain is $[3, \infty)$; range is $[0, \infty)$.
- **Representing Functions**
  - Commonly using tables, graphs, or formulas.
  - **Tables**: Frequent in real-world data (Table 1.1).
  - **Graphs**: Visual representation from data points (Figure 1.6, Figure 1.7).
  - **Algebraic Formulas**: Many applications have explicit formulas, e.g., $A = \pi r^2$.
- **Zeros of a Function**
  - x values where $f(x) = 0$, indicating graph's x-axis intersections.
- **y-intercept**
  - Given by $f(0)$, a function has at most one y-intercept.
- **Vertical Line Test**
  - **Rule**: A graph represents a function if any vertical line intersects it at most once.
- **Increasing and Decreasing Functions**
  - **Definition**: f is **increasing** on I if $x_1 < x_2 \implies f(x_1) \leq f(x_2)$ for $x_1, x_2 \in I$.
  - **Definition**: f is **decreasing** on I if $x_1 < x_2 \implies f(x_1) \geq f(x_2)$ for $x_1, x_2 \in I$.
  - **Example**: $f(x) = x^3$ is increasing on $(-\infty, \infty)$; $f(x) = -x + 1$ is decreasing on $(-\infty, \infty)$ (Figure 1.11).
- **Combining Functions**
  - **Mathematical Operators**: Given f, g, new functions are:
    - $(f + g)(x) = f(x) + g(x)$.
    - $(f - g)(x) = f(x) - g(x)$.
    - $(f \cdot g)(x) = f(x)g(x)$.
    - $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$ (if $g(x) \neq 0$).
  - **Function Composition**: Applying one function to another.
    - **Definition**: For $f : D_f \to R_f$ and $g : D_g \to R_g$, if $R_f \subseteq D_g$, then $(g \circ f)(x) = g(f(x))$ is the composite function with domain D_f.
    - **Example 1.7**: For $f(x) = x^2, g(x) = \sqrt{x-3}$:
      - $(g \circ f)(x) = \sqrt{x^2 - 3}$. Domain: $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$.
      - $(f \circ g)(x) = x - 3$. Domain: $[3, \infty)$.
- **Symmetry of Functions**
  - **Even Function**: $f(x) = f(-x)$ for all x in domain. Symmetric about y-axis.
    - **Example**: $f(x) = x^2$ is even.
  - **Odd Function**: $f(-x) = -f(x)$ for all x in domain. Symmetric about the origin.
    - **Example**: $f(x) = x^3$ is odd.
  - **Example 1.10**:
    - $f(x) = 3x^4 - 1$: Even.
    - $f(x) = \sqrt{x+1}$: Neither.
- **Absolute Value Function**
  - **Definition**: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.
  - It is an even function, symmetric about the y-axis (Figure 1.14).

1.2 | Basic Classes of Functions

This section covers linear, quadratic, and higher-degree polynomials, distinguishing algebraic from transcendental functions, and exploring function transformations.

- **Linear Functions and Slope**
  - **Definition**: A **linear function** is $f(x) = mx + b$ (m, b are constants).
  - **Slope**: Change in y per unit change in x, indicating steepness and direction.
    - **Definition**: Slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ for points (x_1, y_1) and (x_2, y_2).
    - In $f(x) = mx + b$, m is the slope, b is the y-intercept.
  - **Point-slope equation**: $y - y_1 = m(x - x_1)$.
  - **Slope-intercept form**: $y = mx + b$.

- **Standard form**: $Ax + By = C$ (A, B not both zero).
- **Example 1.12**: Line through $(-3, 4)$ and $(2, -1)$:
  - Slope $m = -1$.
  - Point-slope: $y - 4 = -1(x - (-3))$.
  - Slope-intercept: $y = -x + 1$.

- **Polynomials**
  - **Definition**: $f(x) = a_n x^n + \cdots + a_0$ where n is a non-negative integer, $a_n \neq 0$.
  - **Degree**: n (highest power).
  - **Leading coefficient**: a_n.
  - **Types**: Constant ($n = 0$), Linear ($n = 1$), Quadratic ($n = 2$), Cubic ($n = 3$).
- **Power Functions**: Form $f(x) = x^n$ (n is a positive integer).
  - If n is even, $f(x) = x^n$ is even (Figure 1.18(a)).
  - If n is odd, $f(x) = x^n$ is odd (Figure 1.18(b)).
- **Behavior at Infinity (End Behavior)**: What $f(x)$ does as $x \to \pm\infty$. For polynomials, the leading term dictates this behavior.
- **Zeros of Polynomial Functions**
  - **Quadratic Formula**: For $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
    - $b^2 - 4ac > 0$: two real solutions.
    - $b^2 - 4ac = 0$: one real solution.
    - $b^2 - 4ac < 0$: no real solutions.
  - **Example 1.14**: For $f(x) = -2x^2 + 4x - 1$, zeros are $x = 1 \pm \frac{\sqrt{2}}{2}$.
- **Algebraic Functions**
  - **Definition**: Involves $+, -, \times, \div$, rational powers, and roots.
  - **Rational Function**: Quotient of two polynomials, $f(x) = \frac{p(x)}{q(x)}$.
    - **Example**: $f(x) = \frac{x+1}{x-1}$.
  - **Root Function**: $f(x) = x^{1/n}$ ($n > 1$ positive integer).
    - **Example**: $f(x) = \sqrt{x}$ (square root), $f(x) = \sqrt{x}$ (cube root).
- **Transcendental Functions**
  - **Definition**: Cannot be described by basic algebraic operations.
  - **Examples**: Trigonometric, exponential, and logarithmic functions.
  - **Example 1.18**: $f(x) = \frac{1}{x^2+1}$ (algebraic); $f(x) = \cos x$ (transcendental).
- **Piecewise-Defined Functions**
  - Functions defined by different formulas in different domain parts. Absolute value function is an example.
  - **Example 1.19**: Graphing $f(x) = \begin{cases} x + 1 & \text{if } x < 2 \\ (x-2)^2 + 3 & \text{if } x \geq 2 \end{cases}$.
- **Transformations of Functions**
  - **Vertical shift**: $f(x) \pm c$ (up/down c units).
  - **Horizontal shift**: $f(x \pm c)$ (left/right c units).
  - **Vertical scaling**: $c \cdot f(x)$ (stretch if $c > 1$, compress if $0 < c < 1$).
  - **Horizontal scaling**: $f(cx)$ (compress if $c > 1$, stretch if $0 < c < 1$).
  - **Reflection**: $-f(x)$ (about x-axis); $f(-x)$ (about y-axis).
  - **Order of Transformations**: Horizontal shift $\to$ Horizontal scaling $\to$ Vertical scaling $\to$ Vertical shift.
  - **Example 1.21**: Graph $f(x) = -\sqrt{x+3} - 3$ by shifting $f(x) = \sqrt{x}$ left 3, reflecting over x-axis, then shifting down 3.

1.3 | Trigonometric Functions

Trigonometric functions model cyclical phenomena. This section defines the six basic functions, their identities, and graph properties.

- **Radian Measure**
  - **Definition**: Arc length on the unit circle corresponding to an angle. 1 radian corresponds to an

arc length of 1.
  - **Conversion**: $360° = 2\pi$ rad; $180° = \pi$ rad.
  - **Example 1.22**:
    * $45° = \frac{\pi}{4}$ rad.
    * $\frac{3\pi}{2}$ rad $= 270°$.
- **Six Basic Trigonometric Functions**
  - **Unit Circle Definition**: For point (x, y) on unit circle with angle θ:
    * $\sin\theta = y$
    * $\cos\theta = x$
    * $\tan\theta = y/x$ $(x \neq 0)$
    * $\csc\theta = 1/y$ $(y \neq 0)$
    * $\sec\theta = 1/x$ $(x \neq 0)$
    * $\cot\theta = x/y$ $(y \neq 0)$.
  - **Right Triangle Definition**: For acute angle θ, opposite o, adjacent a, hypotenuse h:
    * $\sin\theta = o/h$
    * $\cos\theta = a/h$
    * $\tan\theta = o/a$.
  - **Example 1.23**: $\sin(\frac{\pi}{2}) = 1$.
- **Trigonometric Identities**
  - Equations true for all defined angles.
  - **Reciprocal identities**: e.g., $\tan\theta = \sin\theta/\cos\theta$, $\csc\theta = 1/\sin\theta$.
  - **Pythagorean identities**: $\sin^2\theta + \cos^2\theta = 1$, etc..
  - **Addition/Subtraction formulas**: e.g., $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$.
  - **Double-angle formulas**: e.g., $\sin(2\theta) = 2\sin\theta\cos\theta$.
  - **Example 1.25**: Solving $\sin(2x) = \sin x$ yields $x = n\pi$, $x = \frac{\pi}{3} + 2n\pi$, $x = \frac{5\pi}{3} + 2n\pi$.
- **Graphs and Periods of Trigonometric Functions**
  - **Periodic function**: Values repeat.
  - **Period**: Smallest $P > 0$ where $f(x + P) = f(x)$.
  - Periods: $\sin, \cos, \sec, \csc$ are 2π; $\tan, \cot$ are π.
  - **Transformations**: $f(x) = A\sin(B(x - C)) + D$.
    * A: amplitude.
    * B: affects period (Period $= 2\pi/|B|$).
    * C: phase shift (horizontal).
    * D: vertical shift.

1.4 | Inverse Functions

Inverse functions reverse a function's operation. This section defines them, states existence conditions, outlines how to find them, and discusses inverse trigonometric functions.

- **Existence of an Inverse Function**
  - **Definition**: For $f : D_f \to R_f$, its inverse $f^{-1} : R_f \to D_f$ (if exists) satisfies $y = f(x) \iff x = f^{-1}(y)$. f^{-1} is "f inverse".
  - A function needs to be **one-to-one** to have an inverse.
  - **One-to-one function**: $x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.
- **Horizontal Line Test**
  - **Rule**: A function is one-to-one if every horizontal line intersects its graph at most once.
  - **Example 1.28**: $f(x) = \sin x$ is not one-to-one; $f(x) = x^3$ is one-to-one.
- **Finding a Function's Inverse**
  1. Solve $y = f(x)$ for x in terms of y.
  2. Interchange x and y, write as $y = f^{-1}(x)$.
  - **Example 1.29**: For $f(x) = x^3 - 4$:
    * $x = \sqrt{y + 4}$.
    * $f^{-1}(x) = \sqrt{x + 4}$. Domain/Range: $(-\infty, \infty)$ for both f, f^{-1}.
- **Graphing Inverse Functions**

- Graphs of f and f^{-1} are symmetric about the line $y = x$ (Figure 1.39).
- **Restricting Domains**
  - If f is not one-to-one, restrict its domain to a subset where it is one-to-one to define an inverse.
  - **Example**: $f(x) = x^2$ (not one-to-one). Restrict to $[0, \infty)$, inverse is $g^{-1}(x) = \sqrt{x}$.
- **Inverse Trigonometric Functions**
  - Trigonometric functions are periodic, so domains are restricted to define inverses.
  - **Definitions**:
    * $\arcsin x = y \iff \sin y = x$. Domain: $[-1, 1]$, Range: $[-\frac{\pi}{2}, \frac{\pi}{2}]$.
    * $\arccos x = y \iff \cos y = x$. Domain: $[-1, 1]$, Range: $[0, \pi]$.
    * $\arctan x = y \iff \tan y = x$. Domain: $(-\infty, \infty)$, Range: $(-\frac{\pi}{2}, \frac{\pi}{2}]$.
  - **Example 1.32**:
    * $\arcsin(\frac{1}{2}) = \frac{\pi}{6}$.
    * $\cos(\arcsin(\frac{1}{2})) = \frac{\sqrt{3}}{2}$.

1.5 | Exponential and Logarithmic Functions

This section examines exponential and logarithmic functions, their properties for solving equations, the significance of e, and hyperbolic functions.

- **Exponential Functions**
  - **Definition**: $f(x) = b^x$ $(b > 0, b \neq 1)$. Constant base, variable exponent.
  - **Distinction from Power Functions**: $f(x) = x^b$ is a power function.
  - **Graphing Exponential Functions**:
    * Domain: $(-\infty, \infty)$, Range: $(0, \infty)$.
    * If $b > 1$, increasing, $f(x) \to 0$ as $x \to -\infty$ (Figure 1.44).
    * If $0 < b < 1$, decreasing, $f(x) \to 0$ as $x \to \infty$ (Figure 1.44).
  - **Laws of Exponents**:
    1. $b^x b^y = b^{x+y}$
    2. $\frac{b^x}{b^y} = b^{x-y}$
    3. $(b^x)^y = b^{xy}$
    4. $(ab)^x = a^x b^x$
    5. $(\frac{a}{b})^x = \frac{a^x}{b^x}$.
  - **Example 1.33 (Bacterial Growth)**: $P(t) = 100 \cdot 2^{t/3}$. After 3 hours, $P(3) = 200$.
- **The Number e**
  - $e \approx 2.718282$ is the limit of $(1 + \frac{1}{n})^n$ as $n \to \infty$.
  - **Natural Exponential Function**: $f(x) = e^x$. Its graph has slope 1 at $x = 0$ (Figure 1.45).
  - **Example 1.35 (Compounding Interest)**: $A(t) = Pe^{rt}$. Investment of $1000at0.05$ continuous interest: $A(t) = 1000e^{0.05t}$.
- **Logarithmic Functions**
  - **Definition**: Inverse of exponential function $f(x) = b^x$. $y = \log_b x \iff x = b^y$ $(b > 0, b \neq 1)$.
  - **Example**: $\log_{10} 100 = 2$ because $10^2 = 100$.
  - **Natural Logarithm**: Base e, denoted $\ln x$.
  - **Inverse Relationship**: $b^{\log_b x} = x$ and $\log_b(b^x) = x$.
  - **Graphs**: $y = \log_b x$ and $y = b^x$ are symmetric about $y = x$ (Figure 1.46).
  - **Properties of Logarithms**:
    1. $\log_b(xy) = \log_b x + \log_b y$
    2. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$
    3. $\log_b(x^r) = r \log_b x$.
  - **Change-of-Base Formulas**:
    1. $\log_b x = \frac{\log_a x}{\log_a b}$
    2. $\log_b x = \frac{\ln x}{\ln b}$.
  - **Example 1.36**: Solving $e^{5x-3} = 10$ yields $x = \frac{3+\ln 10}{5}$.
  - **Example 1.37**: Solving $\ln(x) + \ln(x-1) = 0$ yields $x = \frac{1+\sqrt{5}}{2}$ (for $x > 1$).

  - **Example 1.39 (Richter Scale)**: $M_1 - M_2 = \log_{10}(\frac{A_1}{A_2})$. A magnitude 9 earthquake is ≈ 50.11 times more intense than a magnitude 7.3 earthquake.
- **Hyperbolic Functions**
  - **Definition**: Defined by combinations of e^x and e^{-x}.
    * $\cosh x = \frac{e^x + e^{-x}}{2}$
    * $\sinh x = \frac{e^x - e^{-x}}{2}$
    * $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.
  - **Identities**: $\cosh^2 x - \sinh^2 x = 1$.
- **Inverse Hyperbolic Functions**
  - Inverses of hyperbolic functions, expressed using natural logarithms.
  - **Definition**:
    * $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$.
    * $\cosh^{-1} x = \ln(x + \sqrt{x^2-1})$ $(x \geq 1)$.
    * $\tanh^{-1} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$ $(|x| < 1)$.
  - **Example 1.41**: $\sinh^{-1}(1) = \ln(1 + \sqrt{2})$.